Poznan University of Technology

Computer simulation of thermal convection in Rayleigh-Bénard cell

Symulacja komputerowa konwekcji cieplnej w komórce Rayleigha-Bénarda

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Chapter 1: Abstracts

1.1. Abstract

In this work the thermal convection phenomena, which occur in the two-dimensional thin layer of fluid of infinite length. The layer is heated from below and simultaneously cooled at the top as a result the particles of the fluid begin to move creating convectional rolls. These phenomena are known as Rayleigh-Bénard convection. This problem is fully described by the couple of partial differential equations i.e. the Navier-Stokes equation and the thermal diffusion equation. These equations were transformed into the system of three ordinary differential equations well-known as the Lorenz model. This model is very useful in studying the chaotic behaviour of the fluid which occurs in described phenomena. The Lorenz model was used to carry out the computer simulation of convection showing the fluid behaviour with respect to different parameters. The results were compared with the simulation form the fluid dynamics program. Because of the fact that numerical calculation is never precise there was the analysis made which shows how the accuracy of calculation changes the result. Finally the pattern formation in convective fluid is described. This behaviour is characteristic for self-organized systems which manifest the ordered structure in the state far from equilibrium.

1.2. Polish abstract (streszczenie)

W pracy zaprezentowane zostało zjawisko konwekcji cieplnej zachodzącej w cienkiej dwuwymiarowej warstwie płynu, o nieskończonej długości. Warstwa ta jest podgrzewana od dołu i jednocześnie chłodzona od góry, w efekcie cząsteczki płynu zaczynają się poruszać tworząc tzw. rolki konwekcyjne. Zjawisko to powszechnie znane jest jako konwekcja Rayleigha-Bénarda. Układ Opisany jest przy użyciu cząstkowych równań różniczkowych: równania Naviera-Stokesa oraz równanie przewodnictwa ciepła. Przekształcenie tych równań oraz dzięki zastosowanie rozwinięcia w szereg Fouriera przy pewnych założeniach prowadzi do modelu opisanego za pomocą trzech zwyczajnych równań różniczkowych znanych powszechnie jako układ Lorenza. Układ ten umożliwia analizę chaotycznego zachowania płynu, jakie zachodzi w omawianym zjawisku. Model Lorenza został wykorzystany do przeprowadzenia symulacji komputerowej konwekcji prezentującej zachowanie się płynu dla różnych parametrów. Wyniki porównano z symulacją konwekcji otrzymaną z programu służącego do analizy dynamiki płynów. Ponieważ obliczenia numeryczne obarczone są zawsze błędem związanym z zastosowaną metodą obliczeń przedstawiony również został wpływ, jaki ma precyzja wykonywanych obliczeń na uzyskane wyniki. Ponadto przedstawione zostało zjawisko tworzenia się regularnego wzoru, które towarzyszy konwekcji Rayleigha-Bénarda. Takie zachowanie charakterystyczne jest dla systemów samoorganizujących się, które zachowują się w sposób uporządkowany będąc w stanie dalekim od równowagi.

Chapter 2: Introduction

2.1. Rayleigh-Bénard Convection

The Rayleigh-Bénard convection is a problem which has been studied for over a century and it's still a very interesting problem for many researchers all over the world, and it's used e.g. in astrophysics, geophysics, and atmospheric sciences [8]. This theory is very useful in describing weather phenomena and long-term climatic effects [11]; consequently there are many applications which are based on this theory, such as Solar Energy systems (e.g. Power Tower) [10], energy storage and material processing. Not only for its practical significance is this problem so important, but also for purely theoretical reasons as well. The Rayleigh-Bénard convection model is an infinite, thin layer of fluid (practically a very long). The fluid is heated from below while the top one stays colder. The temperature gradient is crucial for the problem: if it's below a certain value, the fluid stays stable despite its natural tendency to move because of its viscosity and thermal diffusivity. On the other hand, when the temperature gradient is over the critical value thermal instability occurs. The men who first considered the problem at the beginning of 20^{th} century were Rayleigh and Bénard, the former provided some theoretical basis for the convection phenomena, while the latter executed some experiments in order to demonstrate the onset of thermal instability. The phenomena of thermal convection were called the Rayleigh-Bénard convection in their honour. However, there is a difference between their attitudes to the problem. Bénard's researches concerned the instability caused by the temperature dependence of the surface-tension coefficient whereas Rayleigh was interested in the convection which occurs and arises as a result of temperature and density nonuniformity. At present, the mechanism studied by Rayleigh is called the Rayleigh-Bénard convection while thermocapillary convection is called Bénard-Marangoni convection.

The mathematical model of this problem is set of non-linear coupled partial differential equations and the solution of this set is degenerate and non-unique [2]. This model is an example of a non-linear system and it can provide an insight into nonlinear phenomena studies. Another property of the Rayleigh-Bénard system is its time dependence, which seems to be one of the most important aspects of transition from laminar to turbulent flow [6]. It is believed that the Rayleigh-Bénard system is a very important part of low-dimensional and spatiotemporal chaos theories and, what is more, it's also a canonical example of a continuous system which is able to generate and sustain spatiotemporal chaos [4]. The system is also an example of self-organization (a pattern forming system) which makes it the most carefully studied system of this kind. Particularly synergetic specialists are interested in this system because it's possible to observe some essential features for nonlinear pattern-forming process [9]. Such formations of patterns occur in crystal growth, solidification fronts' propagation instabilities of nematic liquid crystal, buckling of thin plates and shells, etc. It's also possible to observe them in sand ripples on beaches and desert; in geological formations, in interacting laser beams, and instabilities of numerical algorithms.

2.2. Objective and layout

The main objective of this work is to create a simplified model of Rayleigh-Bénard convection using the famous Lorenz differential equations system i.e. *Mathematica*. Having this model done it will be simulated using symbolic algebra system. The simulation will be connected with the discussion of the results. Finally another computer simulation will be done using fluid dynamics software and the result of this simulation will be compared with the previous one.

Firstly the geometry of the Rayleigh-Bénard convection model is presented in the third chapter, also the transition from thermal conduction to convection is shown. The next chapter is the introduction of the dimensionless constant i.e. the Rayleigh number which is an essential parameter of the description of considered model. The fifth chapter contains the full formal description of convection. Standard equations of fluid mechanics and thermal energy diffusion are transformed to the well-known Lorenz model of convection. The chapter number six is the presentation of the computer simulation of the convection. Both Lorenz model of convection, and the CFD software results are shown. In addition to this, small discussion about numerical precision is carried out. The next chapter is the description of the pattern formation phenomenon and the Rayleigh-Bénard convection is presented as an example of the self-organized system. Finally some conclusions are given in the ninth chapter. At the end of the work there are two appendix, where the former is the table of standard fluid properties while the latter is the listing of the program code.

Chapter 3: Description of the geometry in Rayleigh-Bénard model

The Geometry of the Rayleigh-Bénard model is presented below:

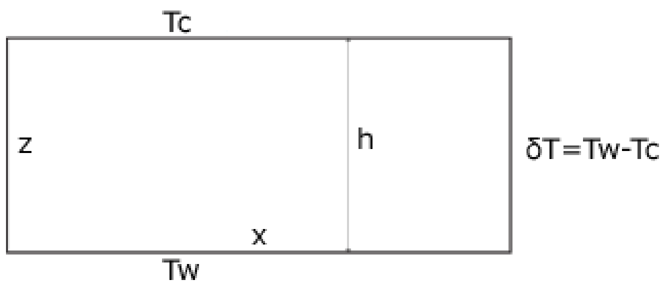


Fig. 3.1: The fluid layer model.

The model is a very long narrow fluid layer. There are fixed temperatures at the top T_C and at the bottom T_w and the temperature at the bottom is higher so $T_w > T_c$. The difference of the temperature is expressed by the term $\delta T = T_w - T_c$ and this is one of the control parameters of the system. Convection appears when the temperature gradient is big enough, consequently a small packet of fluid starts to move up into the colder region of higher density. If the buoyant force caused by difference of density is big enough, then the pocket moves upward so fast that the temperature cannot drop and the convective flow appears. There is also possible that the buoyant force is not strong enough, in such a situation the temperature of the pocket is able to drop before it can move up too much, and as a result fluid stays stable.

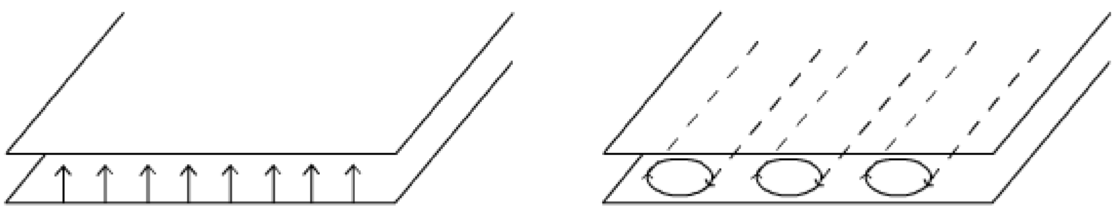


Fig. 3.2: Transition from thermal conduction to convective rolls in infinite two-dimensional fluid layer.

Chapter 4: The Rayleigh number

Using information about thermal energy diffusion and viscous forces in fluid one can study the stability of the fluid [1]. First of all a small pocket of fluid is taken. It moves upward by a small distance Δz so the surrounding temperature is lower by:

$$\Delta T = (\frac{\delta T}{h}\Delta z). \tag{4.1}$$

From the thermal energy diffusion equation one can obtain that the rate of change of temperature is equal to the thermal diffusion coefficient D_T multiplied by the Laplacian of the temperature function. If the displacement is small enough the Laplacian may be approximated by:

$$\nabla^2 T \approx \frac{\delta T}{h^2}\frac{\Delta z}{h}. \tag{4.2}$$

Now the thermal relaxation time δt_T will be defined such that:

$$\delta t_T \frac{dT}{dt} = \Delta T = \delta t_T D_T \nabla^2 T , \tag{4.3}$$

where the second equality follows from the thermal diffusion equation.

Using Laplacian approximation it's obtained that δt_T is:

$$\delta t_T = \frac{h^2}{D_T}. \tag{4.4}$$

The next problem is the effect of the buoyant force on the pocket of fluid which is proportional to the density different between the pocket and its surroundings. On the other hand the density difference is proportional to the thermal expansion coefficient α and the temperature difference ΔT. Consequently the buoyant force may be calculated as:

$$F = \alpha\rho_0 g\Delta T = \alpha\rho_0 g\frac{\delta T}{h}\Delta T \,, \tag{4.5}$$

where: ρ_0 - the original fluid density; g - strength of the local gravitational field.

Assuming that buoyant force balances the fluid viscous force the pocket moves with the constant velocity υ_z .Hence the displacement through a distance Δz takes for the pocket a time:

$$\tau_d = \frac{\Delta z}{\upsilon_z}. \tag{4.6}$$

As the viscous force is equal to the viscosity of the fluid multiplied by the Laplacian of the velocity, the viscous force may be approximated as follows:

$$F_v = \mu\nabla^2\upsilon_z \approx \mu\frac{\upsilon_z}{h^2}. \tag{4.7}$$

Now, by equating buoyant and viscous force one can obtain the υ_z expression:

$$\upsilon_z = \frac{\alpha\rho_0 gh\delta T}{\mu}\Delta z \,, \tag{4.8}$$

and the displacement time:

$$\tau_d = \frac{\mu}{\alpha\rho_0 gh\delta T}. \tag{4.9}$$

If the thermal diffusion time is less than the corresponding displacement time the convection does not appear but if the thermal diffusion time is longer then the fluid pocket will continue to feel an upward force and the convection will continue. The factor which contains the ratio of the thermal diffusion time to the displacement time is the Rayleigh number R and it takes form:

$$R = \frac{\alpha\rho_0 gh^3\delta T}{D_T\mu}. \tag{4.10}$$

The Rayleigh number is the critical parameter for the Rayleigh- Bénard convection.

Chapter 5: Governing equations

There are several methods which could be used to derive the Lorenz equations system, one of them is presented below [5]. The Navier-Stokes equation for fluid flow and thermal energy diffusion equation are used. This problem, like many others, has no exact analytical solution, so approximation methods will be used in order to create a possibly reliable theoretical model. The Lorenz equations system is one of the most famous models in the domain of nonlinear dynamics i.e. it can be applied to describe the motion of the fluid under conditions of Rayleigh-Bénard flow which have been already presented.

As a result of the assumed two-dimensional geometry, only vertical and horizontal velocity components are considered. The form of Navier-Stokes equations for this case is as follows:

$$\begin{aligned} \rho \frac{\partial \upsilon_z}{\partial t} + \rho \vec{\upsilon} \cdot grad\, \upsilon_z &= -\rho g - \frac{\partial p}{\partial z} + \mu \nabla^2 \upsilon_z \\ \rho \frac{\partial \upsilon_x}{\partial t} + \rho \vec{\upsilon} \cdot grad\, \upsilon_x &= -\frac{\partial p}{\partial x} + \mu \nabla^2 \upsilon_x \end{aligned}, \qquad (5.1)$$

where: ρ - mass density of the fluid; g - strength of the local gravitational field; p - fluid pressure; μ - fluid viscosity

The next step is to describe the temperature T. It's done using thermal diffusion equation in the form:

$$\frac{\partial T}{\partial t} + \vec{\upsilon} \cdot grad\, T = D_T \nabla^2 T\ , \qquad (5.2)$$

where D_T - thermal diffusion coefficient.

If the fluid stays stable (there are no convectional phenomena) the temperature changes linearly in accordance with the height (from the bottom to the top):

$$T(x,z,t) = T_w - \frac{z}{h}\delta T \ . \tag{5.3}$$

More important is how the temperature changes when the convection appears so that the relation is not linear anymore. The function which describes temperature deviation from linear is $\theta(x,z,t)$:

$$\theta(x,z,t) = T(x,z,t) - T_w + \frac{z}{h}\delta T \ . \tag{5.4}$$

This function satisfies the following equation:

$$\frac{\partial\theta}{\partial t} + \vec{\upsilon}\cdot grad\,\theta - \upsilon_z\frac{\delta T}{h} = D_t\nabla^2\theta \ . \tag{5.5}$$

Fluid convection is the result of fluid density variation which depends directly on the temperature. The higher the temperature, the density decreases, so a buoyant force appears causing the convection phenomena. The fluid density variation can be described in terms of a power series expansion:

$$\rho(T) = \rho_0 + \frac{\partial\rho}{\partial T}(T - T_w) + \ldots \ , \tag{5.6}$$

where ρ_0 is the fluid density evaluated at T_w

This equation can be presented in another form by introducing the thermal coefficient:

$$\alpha = -\frac{1}{\rho_0}\frac{\partial \rho}{\partial T}. \tag{5.7}$$

Furthermore the expression $(T - T_w)$ in (eq. 6.4) is used thus the equation of the density is as follows:

$$\rho(T) = \rho_0 - \alpha\rho_0[-\frac{z}{h}\delta T + \theta(x,z,t)]. \tag{5.8}$$

There are a few terms of The Navier-Stokes equation in which density ρ occurs, however according to Boussinesq approximation the density variation in may be ignored all terms except the one that involves gravity force [3]. The v_z equation in (eq. 6.1) may be now written by applying this approximation in the following form:

$$\rho_0\frac{\partial \upsilon_z}{\partial t} + \rho_0\vec{\upsilon}\cdot grad\,\upsilon_z = -\rho_0 g - \alpha g\rho_0\frac{z}{h}\delta t - \frac{\partial p}{\partial z} + \alpha g\rho_0\theta(x,z,t) + \mu\nabla^2\upsilon_z. \tag{5.9}$$

If the fluid is stable the first three terms on the right-hand side must add to 0, then an effective pressure gradient is introduced. This gradient is equal to 0 if the fluid is not in motion:

$$\begin{aligned} p' &= p + \rho_0 gz + \alpha g\rho_0\frac{z^2}{2}\frac{\delta T}{h} \\ \frac{\partial p'}{\partial z} &= \frac{\partial p}{\partial z} + \rho_0 g + \alpha g\rho_0\frac{z}{h}\delta T \end{aligned}. \tag{5.10}$$

Now the effective pressure gradient is applied to the Navier-Stokes equations which are simultaneously divided through by ρ_0:

$$\begin{aligned} \frac{\partial \upsilon_z}{\partial t} + \vec{\upsilon} \cdot grad\, \upsilon_z &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \alpha\theta g + \nu\nabla^2\upsilon_z \\ \frac{\partial \upsilon_x}{\partial t} + \vec{\upsilon} \cdot grad\, \upsilon_x &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu\nabla^2\upsilon_x \end{aligned}, \qquad (5.11)$$

where $\nu = \frac{\mu}{\rho_0}$ - kinematic viscosity

5.1. Introducing dimensionless variables

Now some dimensionless variables will be introduced in order to make the system much easier to study. This procedure is very important for seeing which combination of parameters is more important that the others.

The new dimensionless time variable t' is introduced:

$$t' = \frac{D_T}{h^2} t \quad , \tag{5.12}$$

where the expression $\frac{D_T}{h^2}$ is a typical thermal diffusion time over the distance h.

Distance variables x', z':

$$\begin{aligned} x' &= \frac{x}{h} \\ z' &= \frac{z}{h} \end{aligned} \quad . \tag{5.13}$$

Temperature variable θ':

$$\theta' = \frac{\theta}{\delta T} \quad . \tag{5.14}$$

Having these variables defined, it's also possible to introduce a dimensionless velocity:

$$\begin{aligned} \upsilon_x' &= \frac{dx'}{dt'} = \frac{D_T}{h^2} \upsilon_x \\ \upsilon_z' &= \frac{dz'}{dt'} = \frac{D_T}{h^2} \upsilon_z \end{aligned} \quad . \tag{5.15}$$

Then the new form of the Laplacian as follows:

$$\nabla'^2 = h^2 \nabla^2 \ . \tag{5.16}$$

The next step is to put these variables into the Navier-Stokes equation and then myltiply through by $\frac{h^3}{\nu D_T}$:

$$\begin{aligned} \frac{D_T}{\nu}\left[\frac{\partial \upsilon'_z}{\partial t'} + \vec{\upsilon}\cdot grad'\,\upsilon'_z\right] &= -\frac{h^2}{\nu D_T \rho_0}\frac{\partial p'}{\partial z'} + \frac{\alpha \delta T g h^3}{\nu D_T}\theta' + \nabla'^2\,\upsilon'_z \\ \frac{D_T}{\nu}\left[\frac{\partial \upsilon'_x}{\partial t'} + \vec{\upsilon}\cdot grad'\,\upsilon'_x\right] &= -\frac{h^2}{\nu D_T \rho_0}\frac{\partial p'}{\partial x'} + \nabla'^2\,\upsilon'_x \end{aligned} \ . \tag{5.17}$$

Now some of the dimensionless ratios can be replaced with well-known parameters.

Prandtl number σ:

$$\sigma = \frac{\nu}{D_T} \ . \tag{5.18}$$

Rayleigh number R:

$$R = \frac{\alpha g h^3}{\nu D_T}\delta T \ . \tag{5.19}$$

And the last parameter – a dimensionless pressure variable Π:

$$\Pi = \frac{p' h^2}{\nu D_T \rho_0} \ . \tag{5.20}$$

Now the final form of Navier-Stokes and thermal diffusion equations is as falows:

$$\begin{aligned}
&\frac{1}{\sigma}\left[\frac{\partial \upsilon'_z}{\partial t'} + \vec{\upsilon}' \cdot grad\, \upsilon'_z\right] = -\frac{\partial \Pi}{\partial z} + R\theta + \nabla'^2 \upsilon'_z \\
&\frac{1}{\sigma}\left[\frac{\partial \upsilon'_x}{\partial t'} + \vec{\upsilon}' \cdot grad\, \upsilon'_x\right] = -\frac{\partial \Pi}{\partial x} + \nabla'^2 \upsilon'_x \quad . \\
&\frac{\partial \theta}{\partial t} + \vec{\upsilon}' \cdot grad\, \theta - \upsilon'_z = \nabla'^2 \theta
\end{aligned} \tag{5.21}$$

Since now primes will not be written but it's important to remember that they are still there.

5.2. The streamfunction representation of equations

The Streamfunction is the kind of expression which includes the information about all fluid particles motion. The velocity of the fluid flow consists of two components which are the partial derivatives of the streamfunction:

$$\begin{aligned} \upsilon_x &= -\frac{\partial \Psi(x,z,t)}{\partial z} \\ \upsilon_z &= \frac{\partial \Psi(x,z,t)}{\partial x} \end{aligned} \quad . \tag{5.22}$$

The thermal diffusion equation expressed in terms of the streamfunction:

$$\frac{\partial \theta}{\partial t} - \frac{\partial \Psi}{\partial z}\frac{\partial \theta}{\partial x} + \frac{\partial \Psi}{\partial x}\frac{\partial \theta}{\partial z} - \frac{\partial \Psi}{\partial x} = \nabla^2 \theta \ . \tag{5.23}$$

The fluid flow equations:

$$\begin{aligned} \frac{1}{\sigma}\left[\frac{\partial^2 \Psi}{\partial t \partial x} - \frac{\partial \Psi}{\partial z}\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial z \partial x}\right] &= -\frac{\partial \Pi}{\partial z} + R\theta + \nabla^2 \frac{\partial \Psi}{\partial x} \\ \frac{1}{\sigma}\left[-\frac{\partial^2 \Psi}{\partial t \partial z} + \frac{\partial \Psi}{\partial z}\frac{\partial^2 \Psi}{\partial z \partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial z \partial x}\right] &= -\frac{\partial \Pi}{\partial z} - \nabla^2 \frac{\partial \Psi}{\partial z} \end{aligned} \quad . \tag{5.24}$$

Combining these two equations together gives the following result:

$$\begin{aligned} &\frac{1}{\sigma}\left[\frac{\partial}{\partial t}(\nabla^2 \Psi) - \frac{\partial}{\partial z}\left\{\frac{\partial \Psi}{\partial z}\frac{\partial^2 \Psi}{\partial x \partial z} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial z^2}\right\} - \frac{\partial}{\partial x}\left\{\frac{\partial \Psi}{\partial z}\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial z \partial x}\right\}\right] \\ &= R\frac{\partial \theta}{\partial x} + \nabla^4 \Psi \end{aligned} \tag{5.25}$$

Although the pressure term is no longer used, the equation is a complete description of fluid flow.

5.3. Fourier expansion of the streamfunction

In order to solve such partial differential equations, Fourier expansion will be used. According to Fourier's Theorem, every periodic function may be expressed as a sum of a constant term and a series of sine and cosine terms. All the frequencies which are associated with these sines and cosines are integer harmonics of the fundamental frequency. Consequently the solution of the partial differential equation is a product of functions each of which depends on only one of the independent variables (x,z,t). By applying the orthogonalization procedure, the solution is expected to be of the following form [1]:

$$\Psi(x,y,z)= \\ =\sum_{m,n} e^{\omega_{m,n}t}\{A_m\cos(\lambda_m z)+B_m\sin(\lambda_m z)\}\times\{C_n\cos(\lambda_n z)+D_n\sin(\lambda_n z)\}\ , \qquad (5.26)$$

where λs are the wavelengths of the various Fourier spatial mode and ωs are the corresponding frequencies. Such a series may be also expressed as an infinite set of equations. And then the Galerkin technique is used in order to obtain a finite set of equations.

5.4. Boundary Conditions

The boundary conditions for the temperature are as follows:

$$\begin{aligned} z = 0 &\rightarrow \theta = 0 \\ z = 1 &\rightarrow \theta = 0 \end{aligned} \quad . \tag{5.27}$$

It is so because of the fact that the temperature at the top and the bottom is fixed.

Boundary conditions for the streamfunction - let the shear forces at the top and at the bottom be neglected:

$$\begin{aligned} z = 0 &\rightarrow \frac{\partial v_x}{\partial z} = 0 \\ z = 1 &\rightarrow \frac{\partial v_x}{\partial z} = 0 \end{aligned} \quad . \tag{5.28}$$

The following expressions satisfy assumed conditions:

$$\begin{aligned} \Psi(x,z,t) &= \psi(t)\sin(\pi z)\sin(ax) \\ \theta(x,z,t) &= T_1(t)\sin(\pi z)\cos(ax) - T_2(t)\sin(2\pi z) \end{aligned} \quad , \tag{5.29}$$

where the parameter a is to be determined.

The function Ψ is this part of model which is responsible for arising convective rolls which can be observed in real experiment. The second equation is the temperature deviation function which consists of two parts. The former part T_1 describes the temperature difference between the upward and downward moving parts of a convective cell, while the latter is the description of the deviation from the linear temperature variation in the centre of a convective cell as a as a function of vertical position z.

5.5. The Lorenz model of convection

By substituting the assumed form into the (eqs. 6.23 and 6.25), equations for streamfunction and temperature deviation there are many terms which simplify and disappear:

$$\begin{aligned}
&\nabla^2\Psi = -(a^2+\pi^2)\Psi \\
&\nabla^4\Psi = (a^2+\pi^2)^2\Psi \\
&-\frac{d\psi(t)}{dt}(a^2+\pi^2)\sin(\pi z)\sin(ax) = \\
&\quad -\sigma R T_1(t)\sin(\pi z)\sin(ax) + \sigma(a^2+\pi^2)^2\psi(t)\sin(\pi z)\sin(ax)
\end{aligned} \quad . \tag{5.30}$$

The previous equation is true for all values of x and z only if the coefficients of the sine terms satisfy the following equation:

$$\frac{d\psi(t)}{dt} = \frac{\sigma R}{\pi^2+a^2}T_1(t) - \sigma(\pi^2+a^2)\psi(t) \; . \tag{5.31}$$

The form of the temperature deviation equation looks as follows:

$$\begin{aligned}
&\dot{T}_1\sin(\pi z)\cos(ax) - \dot{T}_2\sin(2\pi z) + (\pi^2+a^2)T_1\sin(\pi z)\cos(ax) \\
&\quad -4\pi^2 T_2\sin(2\pi z) - a\psi\sin(\pi z)\cos(ax) \\
&\qquad = -[\pi\psi\cos(\pi z)\sin(ax)][aT_1\sin(\pi z)\sin(ax)] \\
&\qquad\quad -[a\psi\sin(\pi z)\cos(ax)][\pi T_1\cos(\pi z)\cos(ax)] \\
&\qquad\quad +[\psi\sin(\pi z)\cos(ax)][2\pi T_2\cos(2\pi z)]
\end{aligned} \quad . \tag{5.32}$$

Next coefficients $\dot{T}_1, \dot{T}_2$ are found:

$$\begin{aligned}\dot{T}_1 &= a\psi - (\pi^2 + a^2)T_1 - \pi a\psi T_2 \\ \dot{T}_2 &= \frac{\pi a}{2}\psi T_1 - 4\pi^2 T\end{aligned} \quad . \tag{5.33}$$

Finally some new variables will be introduced in order to simplify the notation, the first of them is new time variable:

$$t'' = (\pi^2 + a^2)t' \quad . \tag{5.34}$$

Using this variable and neglecting again primes, the following expressions are set:

$$\begin{aligned}X(t) &= \frac{a\pi}{(a^2 + \pi^2)\sqrt{2}}\psi(t) \\ Y(t) &= \frac{r\pi}{\sqrt{2}}T_1(t) \\ Z(t) &= \pi r T_2(t) \\ r &= \frac{a^2}{(a^2 + \pi^2)^3}R \\ b &= \frac{4\pi^2}{a^2 + \pi^2}\end{aligned} \quad . \tag{5.35}$$

Having all these parameters defined the Lorenz model can be written in the following form [2]:

$$\begin{aligned}\dot{X} &= \sigma(Y - X) \\ \dot{Y} &= rX - XZ - Y \\ \dot{Z} &= XY - bZ\end{aligned} \quad . \tag{5.36}$$

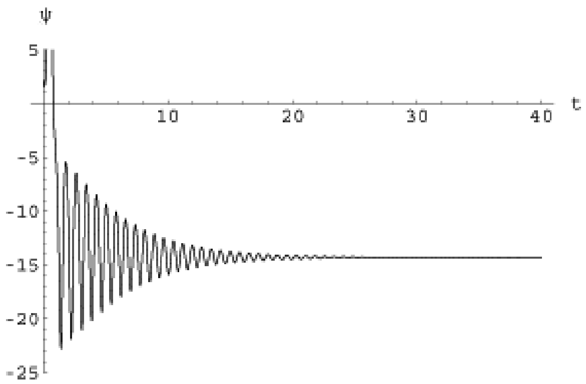
Finally it's important to notice that the truncation of the sine-cosine which was made causes that the Lorenz model concerns only one spatial mode in the x direction with wavelength $\frac{2\pi}{a}$. If the spatial structure of the fluid flow is much more complex or the difference of temperature between top and bottom is too large the Lorenz model is no longer the appropriate description of the fluid dynamics.

Chapter 6: Computer simulation

The simulation of the convection was prepared using *Wolfram Research* software - *Mathematica*. This program is one of the most famous symbolic algebra systems and it is a fully integrated environment for technical computing. The simulation is based on the solution of the system of three ordinary differential equations known as the Lorenz system (5.36). In order to present to the convection phenomena there were maps of temperature generated for certain values of parameters i.e. Rayleigh number, Prandtl number etc. The oscillation and chaotic behaviour are presented using streamfunction spectra plots, the plots of attractors in the phase space and the velocity gradient fields. Finally there was carried out a simulation which shows how important is the precision of the numerical calculation, so that other plots of streamfunction were generated which show the comparison of results obtained with two different working precisions.

6.1. Oscillatory motion

Oscillatory motion is the transitional state of fluid which occurs when the temperature perturbation arises. The particles of fluid begin to move and the behaviour of the fluid seems as if it was convective. Yet the disturbances decrease in a short time and the state of fluid became stable. The simulation was made for the reduced Rayleigh number Rc=18



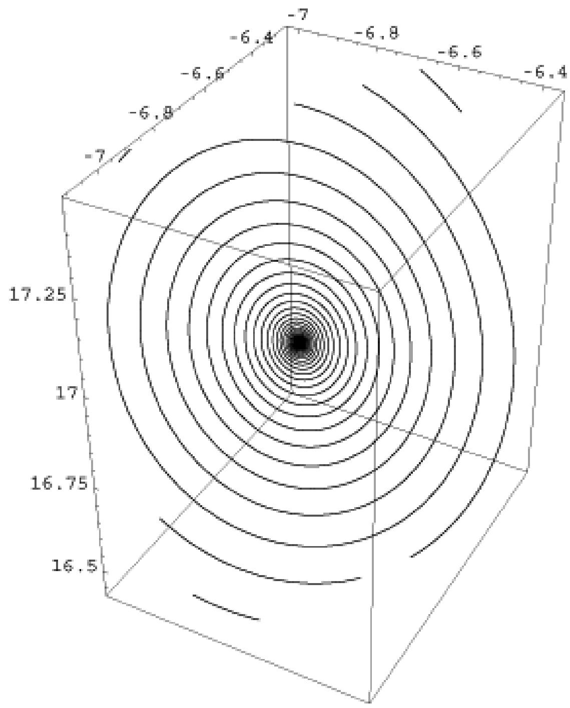
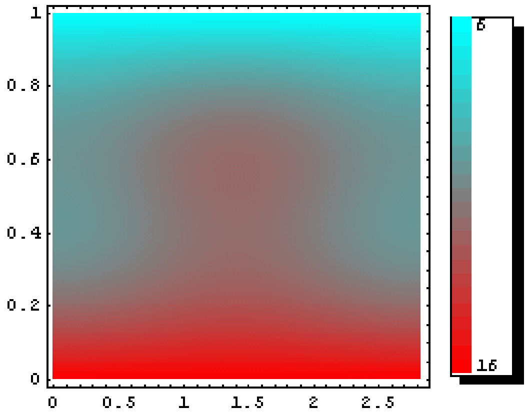


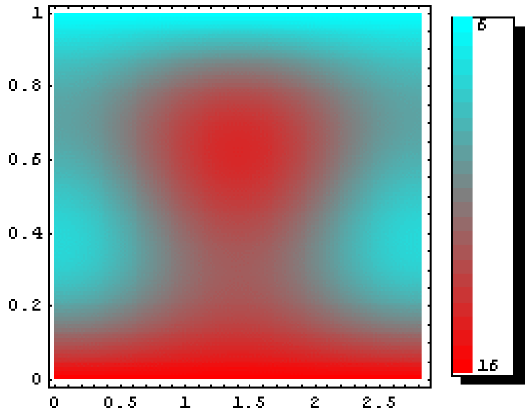
Fig 6.1: The streamfunction plot in the domain of time. The amplitude of oscillations are damping.

Fig 6.2: The plot of the attractor in the phase space

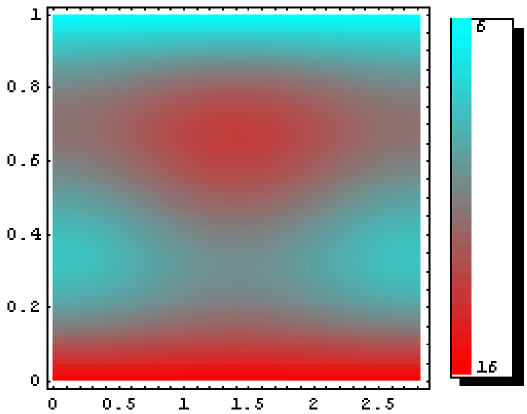
The changes of temperature distribution in the fluid layer are presented below. It starts when the pocket of fluid of higher temperature appears and arises. Next the pocket goes up, spreads and than comes back to the previous state. The process consists of several cycles and after that the fluid becomes stable.



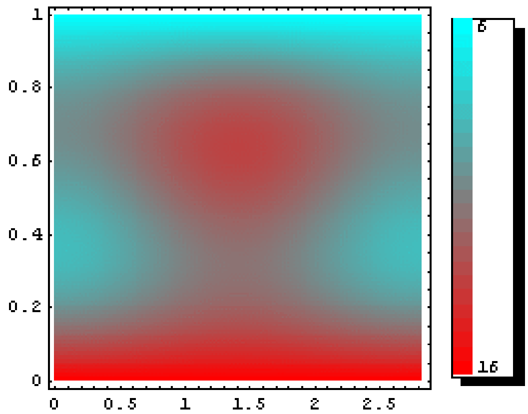
(a) The pocket of the fluid of higher temperature appears



(b) The pocket arises.



(c) The particles of the fluid of higher temperature are spreading a little



(d) The fluid is coming back to the previous state.

Fig 6.3: The sequence of the temperature maps.

6.2. Chaotic behaviour of fluid

The chaotic behaviour of the fluid motion occurs when the reduced Rayleigh number is Rc>24.5, the simulation was made for the reduced Rayleigh number is Rc=28. The result of using the Lorenz model of convection is the characteristic strange attractor which is presented below:

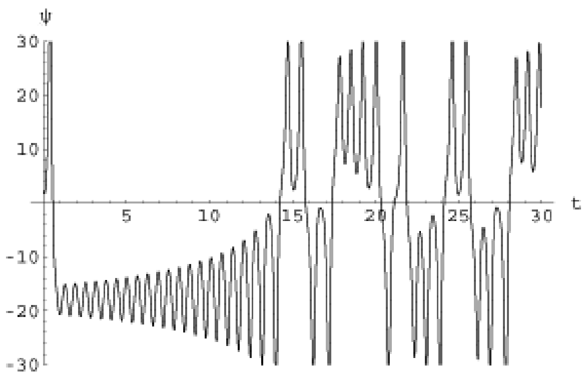


Fig. 6.4: The streamfunction plot in the domain of time. The oscillation increases and the system become non-periodic and consequently chaotic.

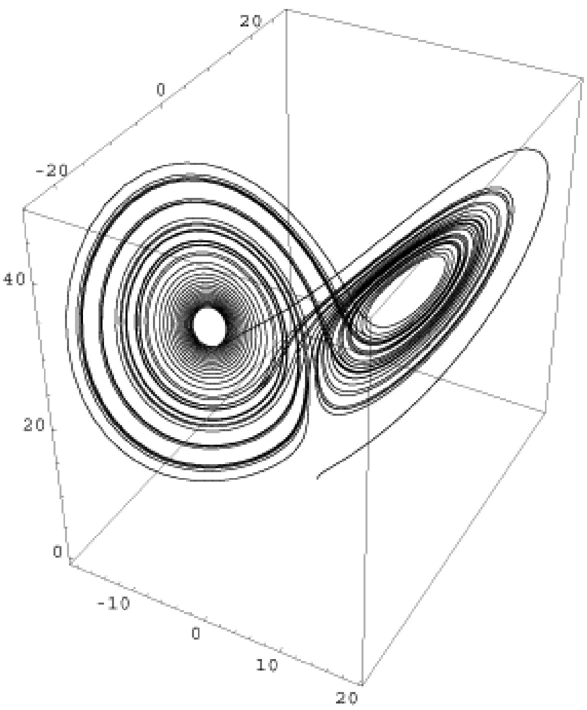


Fig. 6.5. The plot of the strange attractor in the phase space.

Changes of the direction of the gradient of velocity are illustrated below with the plots of the velocity vector fields.

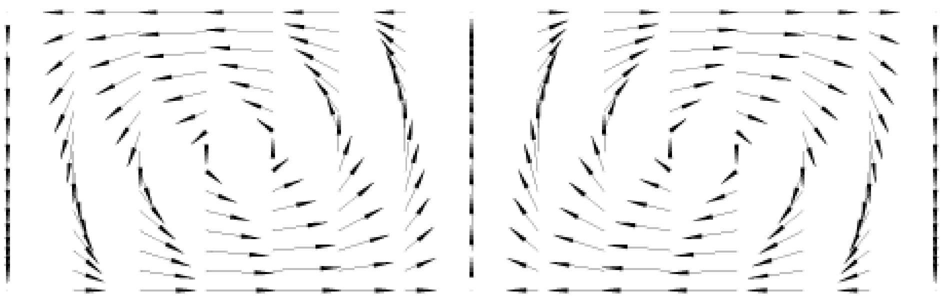


Fig. 6.6: The plot of the velocity vector field at the dimensionless time t=14.1

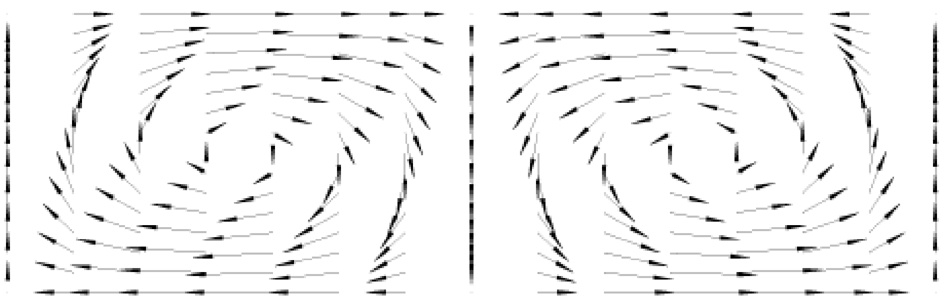
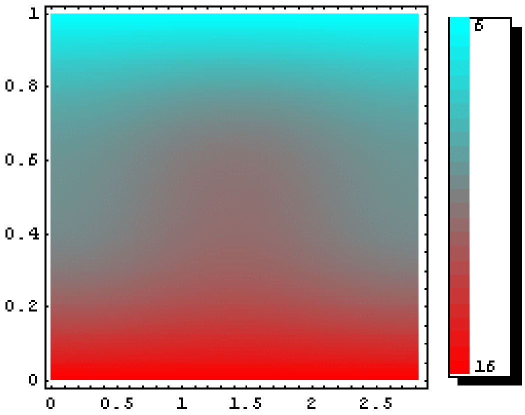
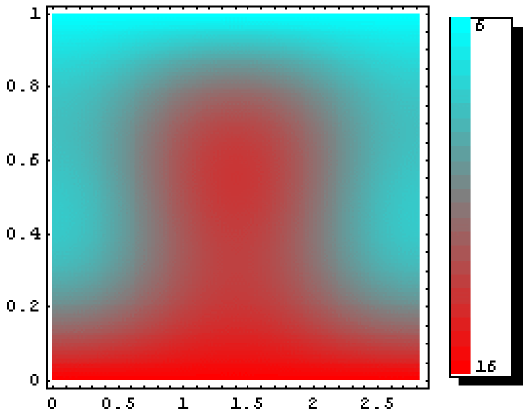


Fig. 6.7: The plot of the velocity vector field at the dimensionless time t=14.3

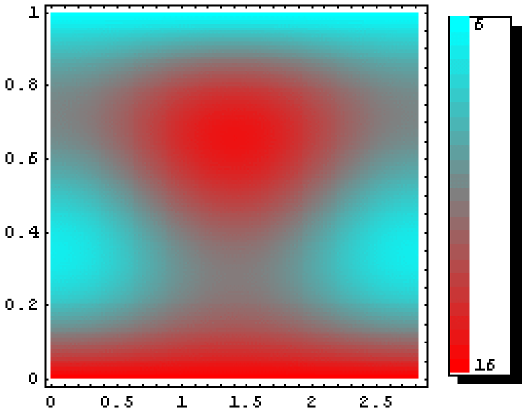
The sequence of the temperature maps was made in the range of the dimensionless time which contains the values used in the plot of the vector filed. The beginning of the process resembles the previous oscillatory motion but after that it's completely different. The state of the fluid doesn't tend to stability but it become non-periodic and the fluid motion switch the direction from one to another. This chaotic behaviour is presented below:



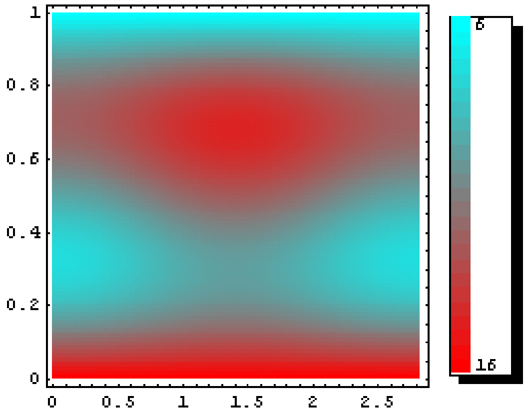
(a) t=13.7



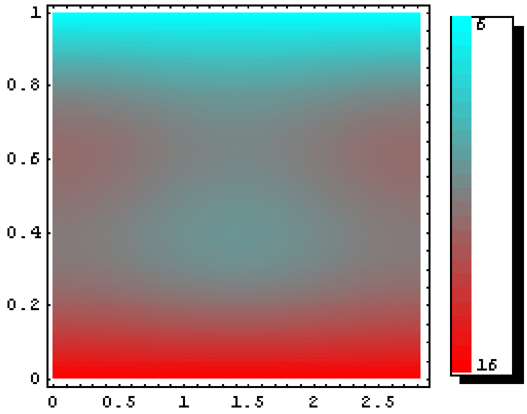
(b) t=13.9



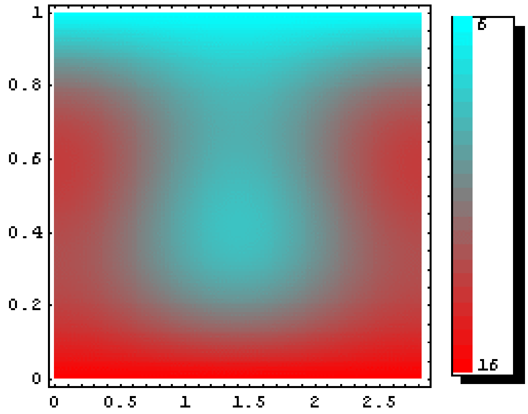
(c) t=14.0



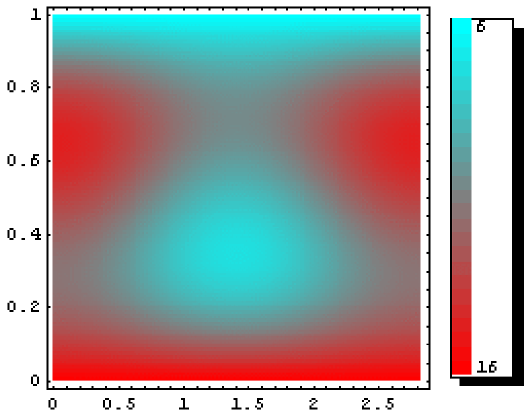
(d) t=14.1



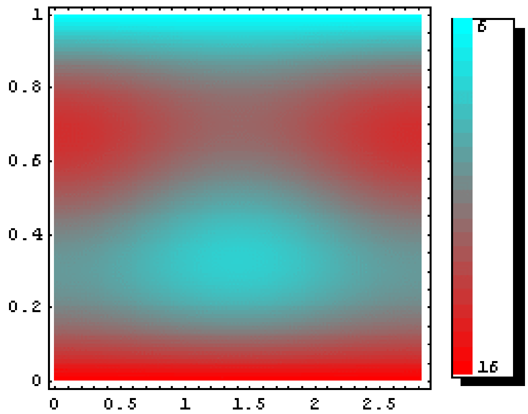
(e) t=14.3



(f) t=14.6



(g) t=14.7



(d) t=14.8

Fig 6.8: the sequence of the temperature maps.

6.3. Numerical accuracy analysis

Nowadays computer simulation is very useful and powerful tool used commonly in a range of researches. Although it is hard to overrate its meaning it is very important to remember that nothing is perfect. Computer is limited by its construction which constrains simulations. Generally the most of calculation are done with the precision which is less or equal to the processor precision. It is impossible to obtain results with a freely high precision so every solution involves an inaccuracy. The second reason why the results are not sufficiently precise is that computer program must usually iterate the same operation for many times with a certain step. In order to increase the accuracy one must decrease the step of iteration so the time of calculation is longer. On the other hand there are many situations in which decreasing the step of iteration below the certain value is pointless because it doesn't change the result much.

Analysis of the chaotic system however is very difficult because of its sensitivity. Consequently every small change in calculation can be the reason of different results. In order to check the influence of the precision of numerical calculation, there were two solutions obtained and they turned out to be different to each other.

The expected difference in results, obtained by solving the problem with two different working precisions, is presented below. The blue line represents the streamfunction which was solved with machine precision whereas the red one is the plot of the solution of higher - 40-digit precision calculation. Although at the beginning both plots are the same, they start to diverge at the dimensionless time ca. t=23.

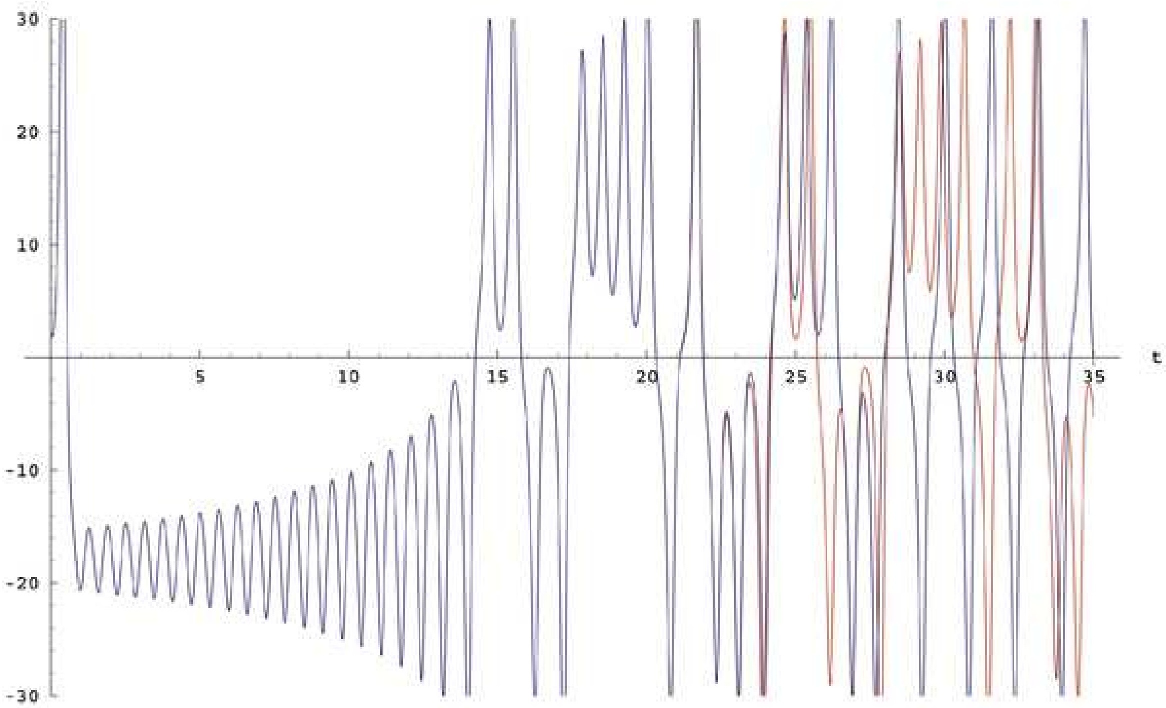
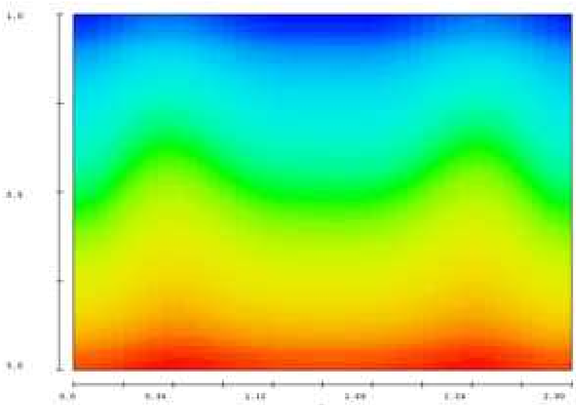


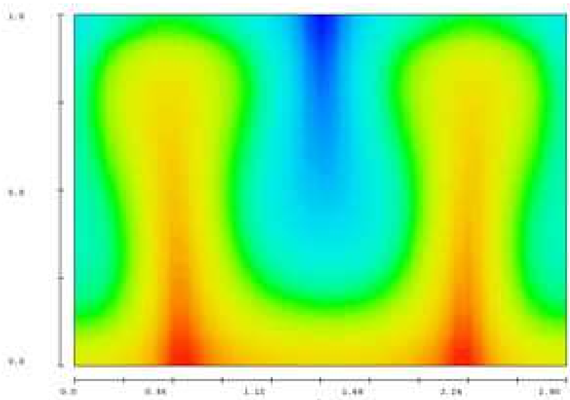
Fig 6.9: The plot of streamfunction solved using different precision.

6.4. CFD program simulation

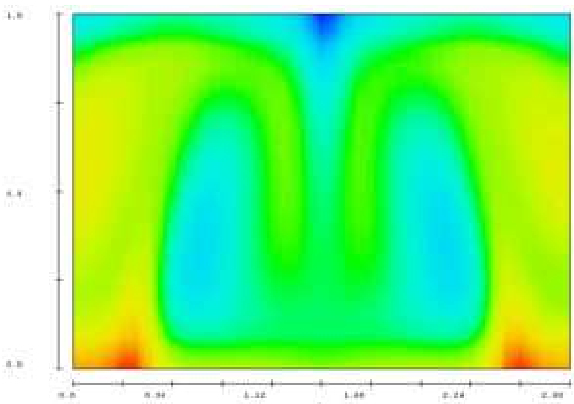
The Computer Fluid Dynamics program was used in order to create another simulation of convection. Air properties were introduced as an input values, the temperature was set as before:16^{o}C at the bottom and 6^{o}C at the top of the fluid layer. The result of the simulation are is follows:



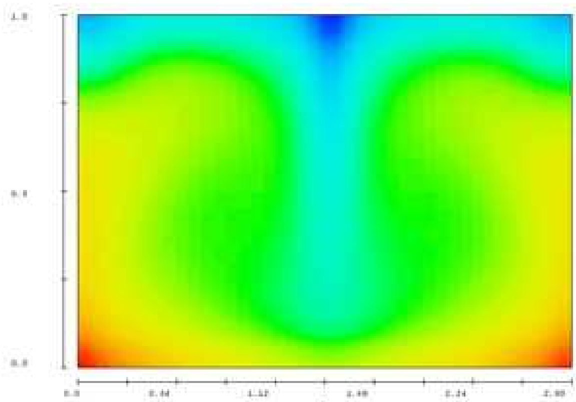
(a) The beginning of the convection



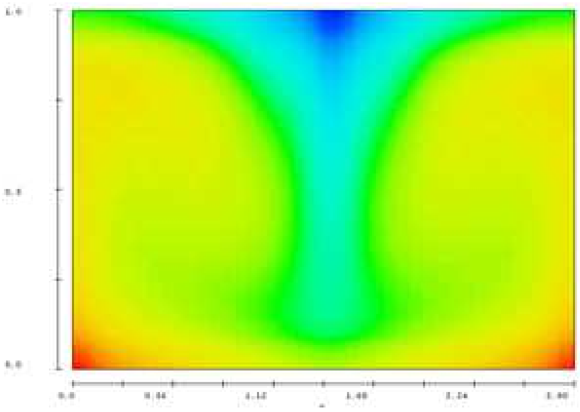
(b)Fluid pockets arises



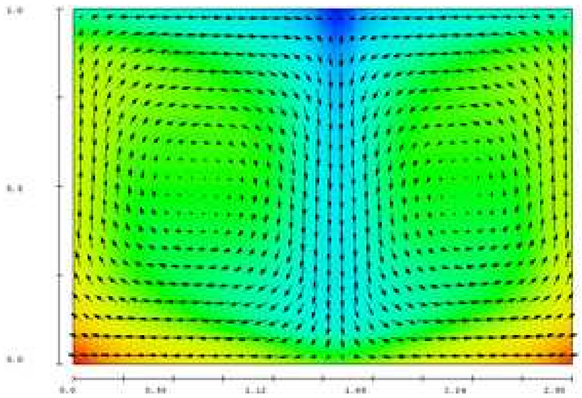
(c) The fluid of greater temperature is spreading.



(d) Convective rolls appear.



(e) Convective rolls.



(f) The temperature map and the velocity vector field.

Fig. 6.10 The sequence of the temperature maps.

Chapter 7: Pattern formation

The Rayleigh-Bénard system evolution is strictly dependent on the temperature difference across the fluid layer. Considering the evolution of the system such as the nondimensional temperature difference is increasing the convection phenomena occurs at some threshold Rayleigh number. There is no fluid flow below this certain value and the heat is transmitted only by conduction through the fluid. With respect to the horizontal walls and having neither special initial conditions nor any variations of the viscosity, the first spatial pattern is found to be a stationary system of parallel rolls. The velocity field of roll convection is nearly two dimensional aside from some usual irregularities or pattern defects.

Although the disturbances which occur at the onset of the convection are described by a particular wave number, the pattern of the convection roll is completely unspecified. It is the result of the fact that a given wave vector can be resolved into two orthogonal components in infinitely many ways. In addition to this, the waves corresponding to different resolutions can be superposed with arbitrary amplitudes and phases. If the space is homogeneous so that there are neither directions nor point preferred in the horizontal plane the entire layer is divided into a mesh of regular polygons with the symmetry planes which are a cell walls [7].

During the experiments two types of pattern are usually observed:

1. Two dimensional rolls which occur when all the quantities depend on only one on the horizontal direction. Then the cells are infinitely elongated so that they can be called rolls instead of cells.
2. Hexagonal cells – they occur when the system is the superposition of three roll sets with wavevectors having the same modulus and direct angle of $\frac{2\pi}{3}$ to one another. There are two variants of this type of cells: l-cells and g-cells and they

are determined by the sign of velocity as a result of increasing or decreasing fluid in the centre of the cell. Mainly g-cells appear in gasses (that is the reason why the name is g-cell) and the l-cells can be observed in liquids (so the name is l-cells).

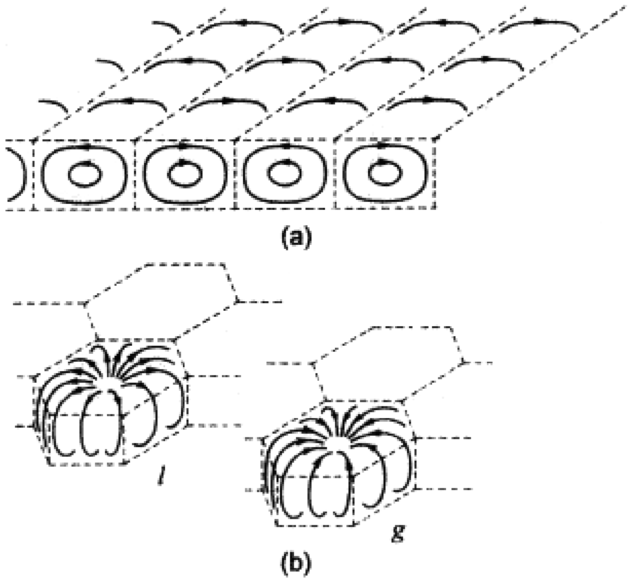


Fig 7.1: Schematic diagram of convection cells (a) two-dimensional rolls. (b) Hexagonal l- and g-cells.

Chapter 8: Conclusions

The results of the simulation show the different kind of behaviour of the fluid flow. The Lorenz model of convection provides both the oscillatory fluid motion of damped amplitude which tends to the stable state, and the non-periodical chaotic fluid behaviour. On the other hand the simulation of convection phenomena were presented using the fluid dynamics software. These results are similar to the previous results, which is a good sign that the Lorenz model can be used as a description of convection when the quite simple example is considered. The Lorenz model is only appropriate for the small Rayleigh numbers when the temperature difference is small enough.

The results of the simulation proved that the precision of the numerical calculation have the significant influence on the solution accuracy and reliability. It is essential part of analysis of chaotic systems.

The Rayleigh-Bénard convection is also an example of self-organization which is a very interesting feature of some chaotic systems. The phenomenon is based on the fact that the system which is far from the equilibrium state manifests highly ordered structure.

Appendix A - Program listing

```
(* Including necessary Mathematica's packages *)

<<Graphics`PlotField`
<<Graphics`Animation`
<<Graphics`Legend`

(* Defining parameters of the equations system *)

(* length *)
L =Sqrt[2];

(* height *)
h =1;

(* wavelength *)
a=π/L;

(* Prandtl number *)
σ=10;

(* Reduced Rayleigh number *)
r=28;

(* geometrical ratio *)
b=4*π^2/(a^2+π^2);

(* temperature at the top *)
Tc=6;

(* temperature at the bottom *)
Tw=16;

(* temperatur difference *)
δT=Tw-Tc;

(* the end of time range *)
endTime=50;

(* the parameters of transformation *)
coeff1=((a^2+π^2)*Sqrt[2])/(a*π);
coeff2=Sqrt[2]/(r*π);
coeff3=1/(π*r);

(* Solving the set of ordinary differential equations, all the computation
will be done with the machine numbers*)
solution=NDSolve[{
      x'[t]==σ*(y[t]-x[t]),
      y'[t]==r*x[t]-x[t] z[t]-y[t],
      z'[t]==x[t] y[t]-b* z[t],
```

```
      x[0]==z[0]==1,
      y[0]==0},
      {x,y,z},
      {t,0,endTime},
      MaxSteps→Infinity,
      WorkingPrecision→MachinePrecision
];
(* this solution will be done with 40-digit presicion *)
solution2=NDSolve[{
      x'[t]==σ*(y[t]-x[t]),
      y'[t]==r*x[t]-x[t] z[t]-y[t],
      z'[t]==x[t] y[t]-b* z[t],
      x[0]==z[0]==1,
      y[0]==0},
      {x,y,z},
      {t,0,endTime},
      MaxSteps→Infinity,

      WorkingPrecision→40
  ];

(*Defining the function to solve the streamfunction *)
Psi[wx_,wz_,t_]:=
    (coeff1*x[t]/.solution)*Sin[π*wz]*Sin[a*wx];

(*Defining the function to solve the temperature deviation *)
Dev[wx_,wz_,t_]:=
    δT^1*((coeff2*y[t]/.solution)*Sin[π*wz]*Cos[a*wx]-
(coeff3*z[t]/.solution)*Sin[2*π*wz]);

(*Defining the function to solve the temperature deviation with higher
precision*)
Dev2[wx_,wz_,t_]:=δT^1*((coeff2*y[t]/.solution2)*Sin[π*wz]*Cos[a*wx]-
(coeff3*z[t]/.solution2)*Sin[2*π*wz]);



(*Defining the function to solve the temperature *)
Theta[wx_,wz_,t_]:=
    Dev[wx,wz,t]+Tw-wz/h*δT;

(*Defining the function to solve the temperature  with higher precision*)
Theta2[wx_,wz_,t_]:=
    Dev2[wx,wz,t]+Tw-wz/h*δT;



(*Defining the function to solve the x-component of velocity vector *)
Vx[wx_,wz_,t_]:=
  Module[{wwx,wwz,tt,res},
    res=-D[Psi[wwx,wwz,tt],wwx];
    res=res/.{wwx→wx,wwz→wz,tt→t};
    Return[res];
```

```
]

(*Defining the function to solve the z-component of velocity vector *)
Vz[wx_,wz_,t_]:=
  Module[{wwx,wwz,tt,res},
    res=D[Psi[wwx,wwz,tt],wwz];
    res=res/.{wwx→wx,wwz→wz,tt→t};
    Return[res];
]

(* Plotting the attractor *)
ParametricPlot3D[
  Evaluate[{x[t],y[t],z[t]}/.sol],
  {t,0,50},
  PlotPoints→5000,
  Boxed→False,
  Axes→False,
  ImageSize→{500,530}
]

(* Plotting the stramfunction*)
Plot[
  Psi[L/2,3/4,s],
  {s,0,30},
  ImageSize→{300,250},
  PlotPoints→5000,
  PlotRange→{-30,30},
  AxesLabel→{"t","ψ"}
]

(* Plotting temperature map *)
TemperaturePlots={}
Do[
  AppendTo[
    TemperaturePlots,
    ShowLegend[
      DensityPlot[
        Theta[wx,wz,s][[1]],
        {wx,0,2*L},
        {wz,0,1},
        ColorFunction→(RGBColor[#,1-#,1-#]&),
        Mesh→False,PlotPoints→100,
        DisplayFunction→Identity,
        ImageSize→{280,280}
      ],
      {RGBColor[#,1-#,1-#]&,25,
      ToString[Tc],
      ToString[Tw],
      LegendPosition→{1.1,-.8},
      LegendSize→{0.3,1.7}}
    ],
  ],
  {s,16,18,0.1}
```

```
];

(* Plotting the velocity vector field *)

PlotVectorField[
  {Vz[wx,wz,14.1][[1]],Vx[wx,wz,14.1][[1]]},
  {wx,0,2 L},
  {wz,0,1},
  AspectRatio→0.3,
  HeadLength→0.02,
  HeadCenter→1,
  HeadWidth→0.2,
  ScaleFunction→(0.003#&),
  ScaleFactor→None
]
```

Appendix B - fluid properties

The tables which are presented below contain some standard values that are used in describing fluids. These properties are necessary to describe fluid flow and they are used to determine the values of some dimensionless numbers. There were two of such numbers determined (assuming that the height of the fluid layer is 1m and the temperature difference between top and bottom is 16 ^{o}C): Rayleigh number, Prandtl number.

Properties of air at 20^{o}C :

Property	**Value**	**Units**
Density	1.2047	$\frac{kg}{m^3}$
Dynamic viscosity	1.8205E-5	$\frac{kg}{m \cdot s}$
Kinematic viscosity	1.5111E-5	$\frac{m^2}{s}$
Thermal diffusion coefficient	2.1117E-5	$\frac{m^2}{s}$
Thermal expansion coefficient	3.4112E-3	$\frac{1}{K}$
Prandtl number	0.71559	
Rayleigh number	1.0487E+6	

Properties of water at 20°C :

Property	**Value**	**Units**
Density	1.2047	$\frac{kg}{m^3}$
Dynamic viscosity	9.7720E-4	$\frac{kg}{m \cdot s}$
Kinematic viscosity	9.7937E-7	$\frac{m^2}{s}$
Thermal diffusion coefficient	1.4868E-7	$\frac{m^2}{s}$
Thermal expansion coefficient	3.4112E-3	$\frac{1}{K}$
Prandtl number	6.5870	
Rayleigh number	2.29814E+9	

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